

Chiral Spin Liquid in Correlated Topological Insulator

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In this paper, we investigate the topological Hubbard model - the spinful Haldane model with on-site interaction on honeycomb lattice with spin rotation symmetry by using slave-rotor approach and find that chiral spin liquid exists in such a correlated electron system of the intermediate coupling region. By considering the anyon nature of excitations, chiral spin liquid may be the ground state of the topological Hubbard model. The low energy physics is basically determined by its Chern-Simons gauge theory.

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The Fermi liquid based view of the electronic properties has been very successful as a basis for understanding the physics of conventional solids including metals and (band) insulators. For the band insulators, due to the energy gap, the charge degree of freedoms are frozen. If there exist spontaneous spin rotation symmetry breaking, the elementary excitations are the gapless spin wave and the gapped quasi-particle (an electron or a hole) that carry both spin and charge quantum numbers. However, in some special insulators, the elementary excitations with fractional quantum numbers of an electron may exist. People call them quantum spin liquid states[1–3]. There exist different types of ansatz of spin liquid: Z_2 , $U(1)$, $SU(2)$ and $SU(2) \times SU(2)$ [2, 3]. These different spin liquid states have the exactly the same global symmetry, as conflicts to Landau's theory, in which two states with the same symmetry belong to the same phase. In particular, there exist quantum spin liquid states breaking time reversal symmetry, of which the elementary excitations are anyons with fractional statistics. People call them *chiral spin liquid (CSL)*[3, 4]. There are two types of CSLs - the abelian CSL with abelian anyonic excitations and non-Abelian CSL with non-Abelian anyons.

Recently, nonAbelian CSL state has been predicted in the Kitaev model on honeycomb lattice or in its generalizations[5]. On the contrary, although Abelian CSL has been proposed much earlier than non-Abelian CSL, till now people don't know any types of model with the (abelian) chiral spin liquid as the ground state. Then one issue here is *may people realize chiral spin liquid in certain many-body systems?* To answer above question we study the quantum properties of the so-called *topological Hubbard model* on honeycomb lattice with spin rotation symmetry by using slave-rotor approach and propose that chiral spin liquid may exist in such a correlated electron system of the intermediate coupling region, of which there exist anyonic excitations.

The topological Hubbard model on honeycomb lattice : The Hamiltonian of the topological Hubbard model on honeycomb lattice is given by

$$H = H_H + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \mu \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}. \quad (1)$$

Here H_H is the spinful Haldane model as

$$H_H = -t \sum_{\langle i,j \rangle, \sigma} \left(\hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + h.c. \right) - t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} e^{i\varphi_{ij}} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma}.$$

t and t' are the real hopping between the first-neighbor and the second-neighbor on the different and the same sublattices, respectively. $e^{i\varphi_{ij}}$ is a complex phase into the second-neighbor hopping, and we set the direction of the positive phase is clockwise ($|\varphi_{ij}| = \frac{\pi}{2}$)[6]. U is the on-site Coulomb repulsion. σ are the spin-indices representing spin-up ($\sigma = \uparrow$) and spin-down ($\sigma = \downarrow$) for electrons. μ is the chemical potential and $\mu = U/2$ when the system is half-filling (in this paper we only study the case of half-filling). $\langle i, j \rangle$ and $\langle\langle i, j \rangle\rangle$ denote two sites on a first-neighbor and a second-neighbor link, respectively. $\hat{n}_{i\uparrow}$ and $\hat{n}_{i\downarrow}$ are the number operators of electrons with up-spin and down-spin respectively. It is obvious that there is spin rotation symmetry but no time reversal symmetry for the topological Hubbard model.

For free fermions (the on-site Coulomb repulsion U is zero), the spectrum is $\mathbf{E}_{\mathbf{k}} = \pm \sqrt{(\xi_{\mathbf{k}})^2 + (\xi'_{\mathbf{k}})^2}$ where $|\xi_{\mathbf{k}}| = t\sqrt{3 + 2\cos(\sqrt{3}k_y) + 4\cos(3k_x/2)\cos(\sqrt{3}k_y/2)}$ and $\xi'_{\mathbf{k}} = 2t' \sum_i \sin(\mathbf{k} \cdot \mathbf{b}_i)$. The parameters $\mathbf{a}_1, \mathbf{a}_2$ and \mathbf{a}_3 are the displacement from one site to its nearest neighborhoods and $\mathbf{b}_1 = \mathbf{a}_2 - \mathbf{a}_3$, $\mathbf{b}_2 = \mathbf{a}_3 - \mathbf{a}_1$, etc. The length of the hexagon side has been chosen to be unit. One can see that there exist an energy gap $\Delta_c = 6\sqrt{3}t'$ at the points $\mathbf{k}_1 = \frac{2\pi}{3}(1, \frac{1}{\sqrt{3}})$ and $\mathbf{k}_2 = \frac{2\pi}{3}(-1, -\frac{1}{\sqrt{3}})$. Due to the existence of nonzero TKNN number[7], there exists the integer quantum Hall effect $\sigma_{xy} = \frac{2e^2}{h}$. Here the parameter 2 comes from the contributions of electrons of up-spin and down-spin. Therefore, for the free fermions, the ground state is a topological insulator with quantized anomalous Hall effect[6, 8, 9].

An issue is whether the topological insulator is stable for the interaction case. To examine stability of the topological insulator against on-site interaction, we will use the slave-rotor approach to study the topological Hubbard model. Slave-rotor approach has been widely applied to study the quantum liquid states near Mott transition of correlated electron systems[10–15]. By the slave-rotor approach, we find that chiral spin liquid appears of the intermediate coupling region.

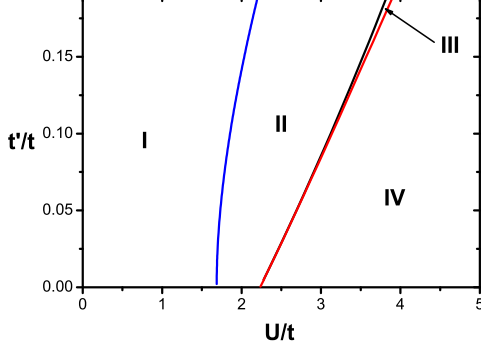


FIG. 1: (color online) Phase diagram at $T = 0$. There are four regions: I is TI state, II is the quantum spin liquid, III is an AF order with QAH effect, IV is the trivial AF order. The blue line and black line are $(\frac{U}{t})_{c1}$ and $(\frac{U}{t})_{c2}$, respectively.

Slave-rotor approach: In slave-rotor approach, we represent electronic operator $\hat{c}_{j\sigma}$ into $\hat{c}_{j\sigma} = e^{i\theta_j} \hat{f}_{j\sigma}$ where the fermion spinon $\hat{f}_{j\sigma}$ represents the spin degree of freedom and $e^{i\theta_j}$ represents the charge degree of freedom together with slave-rotor's constraint $\sum_{\sigma} \hat{f}_{j\sigma}^{\dagger} \hat{f}_{j\sigma} + L_j = 1$.

Here we introduce an additional variable - the angular momentum $L = i\hbar\partial_{\theta}$ associated with a quantum $O(2)$ rotor θ . Then the Hamiltonian in Eq.(1) turns into

$$H_{eff} = -t \sum_{\langle i,j \rangle, \sigma} (\hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} X_i^{\dagger} X_j + h.c.) - \mu \sum_{i, \sigma} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{i\sigma} \quad (2)$$

$$- t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} e^{i\varphi_{ij}} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} X_i^{\dagger} X_j + \frac{U}{2} \sum_i L_i^2$$

$$+ \sum_i h_i (\sum_{\sigma} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{i\sigma} + L_i - 1) + \sum_i \rho_i (|X_i|^2 - 1).$$

where h_i is a Lagrange multiplier for slave-rotor's constraint. ρ_i is a complex Lagrange multiplier for $|X_i|^2 = 1$ ($X_i = e^{i\theta_i}$).

We introduce four variational parameters $Q_f = \langle X_i^{\dagger} X_j \rangle_{ij\langle nn \rangle}$, $Q_X = \langle \sum_{\sigma} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} \rangle_{ij\langle nn \rangle}$, $Q'_f = \langle X_i^{\dagger} X_j \rangle_{ij\langle\langle nn \rangle\rangle}$ and $Q'_X = \langle \sum_{\sigma} e^{i\varphi_{ij}} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} \rangle_{ij\langle\langle nn \rangle\rangle}$. To obtain the five parameters Q_X , Q_f , Q'_X , Q'_f , ρ , we solve the following equations self-consistently,

$$Q_X = \frac{1}{3tN_s} \sum_k \frac{Q_f |\xi_{\mathbf{k}}|^2}{E_f}, \quad Q'_X = \frac{1}{3t'N_s} \sum_k \frac{Q'_f |\xi'_{\mathbf{k}}|^2}{E_f}, \quad (3)$$

$$Q_f = \frac{1}{N_s} \sum_k \frac{|\xi_{\mathbf{k}}|}{12t} \frac{U}{\sqrt{U(\rho + \varepsilon_k)}}, \quad 1 = \frac{1}{N_s} \sum_k \frac{U}{2\sqrt{U(\rho + \varepsilon_k)}},$$

$$Q'_f = \frac{1}{N_s} \sum_k \frac{g_{\mathbf{k}}}{24} \frac{U}{\sqrt{U(\rho + \varepsilon_k)}},$$

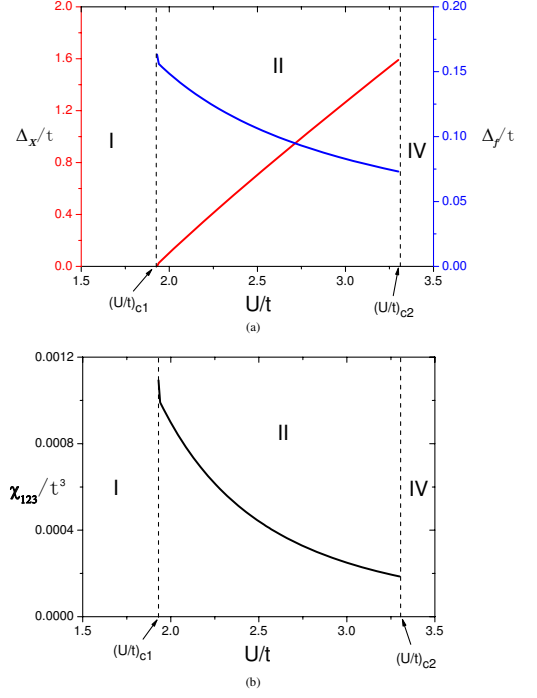


FIG. 2: (color online) (a): The energy gaps of rotor Δ_X (the red line) and fermionic spinon Δ_f (the blue line) of the topological Hubbard model at $T = 0$ and $t' = 0.12$. $(U/t)_{c1}$ and $(U/t)_{c2}$ are the quantum phase transitions. (b) The non-zero spin chiral order parameter.

where

$$g_{\mathbf{k}} = 4 \cos(3k_x/2) \cos(\sqrt{3}k_y/2) + 2 \cos(\sqrt{3}k_y) \quad (4)$$

and $\varepsilon_{\mathbf{k}} = -Q_X |\xi_{\mathbf{k}}| - t' Q'_X g_{\mathbf{k}}$. N_s denoting the number of unit cells.

After the calculation, we find that the topological insulator is stable below a critical interaction strength, $\frac{U}{t} < (\frac{U}{t})_{c1}$ (See the results in Fig.1). With increasing interaction strength, we get non-zero solutions of Q_X , Q_f , Q'_X , Q'_f , ρ . As a result, the ground state turns into a quantum spin liquid state characterized by a finite gap of rotor excitation. The excitations are not fermions generated by \hat{c}_i^{\dagger} , instead, they are rotors and fermionic spinons.

Then we get the mean field effective Hamiltonian as $H_{eff} = H_f + H_X$ where

$$H_f = -t \sum_{\langle i,j \rangle, \sigma} Q_f \hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} - t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} Q'_f e^{i\varphi_{ij}} \hat{f}_{i\sigma}^{\dagger} \hat{f}_{j\sigma} + h.c. \quad (5)$$

and

$$\begin{aligned}
H_X = & -t \sum_{\langle i,j \rangle} Q_X X_i^* X_j - t' \sum_{\langle\langle i,j \rangle\rangle} Q'_X X_i^* X_j + \frac{U}{2} \sum_i L_i^2 \\
& + h \sum_i L_i + \rho \sum_i |X_i|^2 + h.c.
\end{aligned} \quad (6)$$

After diagonalizing the Hamiltonian in Eq.(5), we get the energy spectrum for the two-flavor fermionic spinons $E_f = \pm \sqrt{Q_f^2 |\xi_{\mathbf{k}}|^2 + Q_f'^2 |\xi'_{\mathbf{k}}|^2}$. One may get the energy gap for the fermionic spinons as $\Delta_f = 6\sqrt{3}t'Q'_f$. On the other hand, the spectrum of the charge excitations E_X is $E_X = 2\sqrt{U\rho + U\varepsilon_{\mathbf{k}}}$. In the quantum spin liquid state, Δ_X is not zero, $\Delta_X = 2\sqrt{U(\rho + \min(\varepsilon_{\mathbf{k}}))}$; when approaching the phase transition between topological insulator and quantum spin liquid state it becomes zero due to rotor condensation. From Fig.2.(a), we can notice that the rotor's gap is much larger than the fermionic spinon's gap (the energy scale of Δ_X is 10 times to that of Δ_f).

By this method we get a quantum spin liquid state with fermionic spinons by adding the on-site interaction to the spinful Haldane model. When further increasing the interaction strength, the quantum spin liquid is unstable against antiferromagnetic (AF) spin density wave (SDW) order. Such AF-SDW order is described by $\langle \hat{c}_{i,\sigma}^\dagger \hat{c}_{i,\sigma} \rangle = \frac{1}{2}(1 + (-1)^i \sigma M)$. Here M is the staggered magnetization. In HF mean field approach, we may get the self-consistency equation for M by minimizing the ground state energy. To characterize different orders of the topological Hubbard model (the topological insulator, the quantum spin liquid, the AF-SDW), we plot a phase diagram in Fig.1. $(\frac{U}{t})_{c2}$ denotes another critical interaction strength that divides the quantum spin liquid and the AF-SDW (See Fig.1). In particular, in Fig.1, one may see that there exists a narrow window between quantum spin liquid state and the trivial AF order - an AF-SDW order with quantized anomalous Hall effect.

Effective Chern-Simons theory : In the following parts, we will focus on the quantum spin liquids between topological insulator and AF-SDW state (region II in Fig.1).

In the quantum spin liquid state, because the rotor excitation X_i has a big energy gap, we may integrate out it and concentrate only on the spinon excitations. The fluctuations of h_i and the phase fluctuations of Q_f, Q'_f amount to coupling the fermionic spinons to a compact U(1) gauge field a_{ij} by the minimal prescription. After considering the fluctuations around the mean field saddle point, we get the effective model of fermionic spinons

with U(1) gauge invariance

$$\begin{aligned}
L_f = & \sum_{j,\sigma} \hat{f}_{j\sigma}^* (\partial_\tau - ia_{\tau,j} + h_0 - \mu) \hat{f}_{j\sigma} \\
& - t Q_f \sum_{\langle i,j \rangle, \sigma} e^{ia_{ij}} \hat{f}_{i\sigma}^\dagger \hat{f}_{j\sigma} \\
& - t' Q'_f \sum_{\langle\langle i,j \rangle\rangle, \sigma} e^{i\varphi_{ij}} e^{ia_{ij}} \hat{f}_{i\sigma}^\dagger \hat{f}_{j\sigma} + h.c.
\end{aligned} \quad (7)$$

where $h_0 = \langle h_i \rangle = \mu$. Hence the continuum version of above model becomes the two flavor massive Schwinger model with the Lagrangian as $\mathcal{L}_f = i\bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + m\bar{\psi}\psi$ where $m = \Delta_f/2$ is a fermion mass and $\psi_\alpha = \psi_\alpha^\dagger \gamma_0 = (\bar{f}_{\uparrow\alpha A}, \bar{f}_{\uparrow\alpha B}, \bar{f}_{\downarrow\alpha A}, \bar{f}_{\downarrow\alpha B})$ and $\alpha = 1, 2$ labels the two points $\mathbf{k}_1 = \frac{2\pi}{3}(1, \frac{1}{\sqrt{3}})$ and $\mathbf{k}_2 = \frac{2\pi}{3}(-1, -\frac{1}{\sqrt{3}})$.

Considering the quantum fluctuations of fermionic spinons, we get a two dimensional dynamics Maxwell model of the gauge field a_μ as $\mathcal{L}_a = \frac{1}{4e_a^2}(\partial_\mu a_\nu)^2$ where $e_a^2 = \frac{1}{3\pi} \frac{1}{m}$. The compact U(1) gauge theory is always confining[20]. However, the induced Chern-Simons (CS) term will lead to deconfinement. Integrate over fermions by using $1/m$ (gradient) expansion approach, we obtain the CS term $\mathcal{L}_{cs} = \frac{N}{2} \frac{m}{|m|} \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$ where $N = 4$ (two-flavor plus two spin components)[16, 17].

Finally we obtain an effective CS theory of the quantum spin liquid state with the Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L}_f + \mathcal{L}_a + \mathcal{L}_{CS}$.

Chiral spin liquid : After obtaining the effective CS theory, the quantum spin liquid state (region II in Fig.1) is identified to chiral spin liquid. Such a topologically ordered spin liquid breaks time-reversal symmetry while preserves all other symmetries (spin rotation symmetry, translation symmetry, ...).

Firstly we point out that the quasi-particle is really anyon. Let us consider single π -flux excitation ($\Phi = \pi$) in the quantum spin liquid as shown in Fig.3. The vacuum expectation value of the fermion number $\langle N^f \rangle$ is related to the spectral asymmetry of the Dirac Hamiltonian

$$\langle N^f \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} dE \frac{1}{\pi} \text{Im} \text{Tr} \left(\frac{1}{H_f - E - i\epsilon} \right) \text{sign}(E) \quad (8)$$

where H_f is the Hamiltonian of the fermion spinon[18]. And the fermionic number is also related to the Atiah-Patodi-Singer invariant $\eta_H = -\frac{1}{2}\langle N^f \rangle$ which represents the difference between the number of states with positive and negative energy. The Atiah-Patodi-Singer index theorem states that due to the quantum anomaly the fermionic number of the Dirac operator, equals the topological charge as $\langle N^f \rangle = -\frac{N}{2} \frac{m}{|m|} \frac{\Phi}{2\pi}$, $N = 4$. It represents a fact that a π -flux excitation with half topological charge carries one fermion number, $|\langle N^f \rangle| = 1$. That means a π -flux excitation is really a bound state of π -flux and a fermionic spinon. Due to nontrivial AB phases upon adiabatic exchange of charge and flux, π -flux turns into semion - special type of Abelian anyon. By the fusion

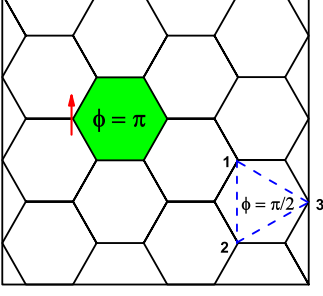


FIG. 3: (color online) Scheme of an anyon as a composite object of a fermionic spinon and a π -flux. 1, 2, 3 correspond three vertices of an equilateral triangle to define the spin chiral order parameter in Eq.(9).

rules of Abelian anyons, one may find a statistical angle θ is $\frac{1}{2} = \frac{\pi}{2}$.

Secondly, we calculate the topological degeneracy of CSL, a topologically ordered spin liquid. In the temporal gauge, $a_0 = 0$, and on a torus, the ground states are characterized by zero momentum gauge fields ($a_{x,\mathbf{k}=0}$, $a_{y,\mathbf{k}=0}$). After straightforwardly calculations[3], we may get the effective Hamiltonian of ($a_{x,\mathbf{k}=0}$, $a_{y,\mathbf{k}=0}$) as $\mathcal{H}_{eff} = \frac{(\mathcal{P}_{\theta_x} - \mathcal{A}_{\theta_y})^2}{2M_x} + \frac{(\mathcal{P}_{\theta_y} - \mathcal{A}_{\theta_x})^2}{2M_y}$ where $\mathcal{A}_{\theta_x} = -\frac{a_{y,\mathbf{k}=0}}{2\pi}$, $\mathcal{A}_{\theta_y} = \frac{a_{x,\mathbf{k}=0}}{2\pi}$ and $M_x = \frac{1}{e_a^2} \frac{L_y}{L_x}$, $M_y = \frac{1}{e_a^2} \frac{L_x}{L_y}$ (L_x and L_y are the lengths of the system along x- and y-directions, respectively). This model corresponds to a particle on a plane with a finite "magnetic field". The strength of the "effective magnetic field" is obtained as $\mathcal{B}_{eff} = \frac{2}{2\pi}$. There exists a two-unit flux tube through the center of the torus. So the degeneracy is given as $\mathcal{D} = 2$.

Next we calculate the edge states of the CSL from the effective CS theory. We know that the charges of a_μ are quantized as integers. Then the effective CS theory has two right-moving edge excitations. The two branches of the edge excitations are described by the following 1D fermion theory $\mathcal{L}_{edge} = \sum_\alpha \psi_{\alpha R}^\dagger (\partial_t - v_R \partial_x) \psi_{\alpha R}$, where $\alpha = 1, 2$. $\psi_{\alpha R}$ carries a unit of a_μ charge. That means we get spin-charge separated edge states: the edge modes

carry only spin current[21, 22].

Thirdly, an important property of CSL is the non-zero spin chiral order parameter which has a non zero expectation value in a phase with broken P and T symmetry. Spin chiral order parameter is a rotationally invariant operator defined through[4, 19]

$$\begin{aligned} \chi_{(123)} &= \langle \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3) \rangle \\ &= \frac{1}{4i} \langle \hat{f}_{1\alpha}^\dagger \hat{f}_{2\alpha} \hat{f}_{2\beta}^\dagger \hat{f}_{3\beta} \hat{f}_{3\gamma}^\dagger \hat{f}_{1\gamma} - \hat{f}_{1\alpha}^\dagger \hat{f}_{3\alpha} \hat{f}_{3\beta}^\dagger \hat{f}_{2\beta} \hat{f}_{2\gamma}^\dagger \hat{f}_{1\gamma} \rangle. \end{aligned} \quad (9)$$

If the sites 1, 2, 3 correspond three vertices of an equilateral triangle in a plaquette, we may estimate the mean field value of $\chi_{(123)}$ to be $\frac{1}{2} (\sin \Phi) |Q'_X|^3$ where Φ is the gauge invariant flux through the equilateral triangle, $\Phi = |\varphi_{12} + \varphi_{23} + \varphi_{31}| = \frac{\pi}{2}$. Thus we get non-zero spin chiral order parameter along these loops. See the results in Fig.2.(b).

In summary, we have predicted an emergent chiral spin liquid state base on the topological Hubbard model on honeycomb lattice with spin rotation symmetry. In the end, we address the relevant experimental realization and the way to be conformed by numerical approaches. In condensed matter physics, there is no such material with a Hamiltonian of the topological Hubbard model. However, such system may be simulated in optical lattice of cold atoms. In Ref.[23, 24], it is proposed that the (spinless) Haldane model on honeycomb optical lattice can be realized in the cold atoms. When two-component fermions with repulsive interaction are put into such optical lattice, one can get an effective topological Hubbard model. It is easy to change the potential barrier by varying the laser intensities to tune the Hamiltonian parameters including the hopping strength (t -term) and the particle interaction (U -term). On the other hand, one may check our prediction by quantum Monte Carlo (QMC) simulations including the global phase diagram, the topological degeneracy, the spin chiral order parameter[4].

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- [1] P. W. Anderson, Science **235**, 1196 (1987).
- [2] X. G. Wen, Phys. Rev. B **65**, 165113 (2002).
- [3] X. G. Wen, *Quantum Field Theory of Many-Body Systems*, (Oxford University Press, 2004).
- [4] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1990).
- [5] A. Kitaev, Ann. Phys. **321**, 2(2006).
- [6] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
- [7] D. J. Thouless, M. Kohmoto, M. P. Nightingale and M.

- den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
- [8] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); **95**, 226801 (2005).
- [9] B. A. Bernevig and S. C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
- [10] Serge Florens, Antoine Georges, Phys. Rev. B **70**, 035114 (2004).
- [11] S. S. Lee and P. A. Lee, Phys. Rev. Lett. **95**, 036403 (2005).
- [12] M. Hermele, Phys. Rev. B **76**, 035125 (2007).
- [13] M. W. Young, S. S. Lee, C. Kallin, Phys. Rev. B **78**,

- 125316 (2008).
- [14] S. Rachel, K. Le Hur, Phys. Rev. B **82**, 075106 (2010).
 - [15] Jeffrey G. Rau, Hae-Young Kee, arXiv:1007.5320.
 - [16] A. N. Redlich, Phys. Rev. Lett. **52** (1984) 18, Phys. Rev. D **29** (1984) 2366.
 - [17] K. Ishikawa and T. Matsuyama, Z. Phys. C **33**, 41 (1986).; Nucl. Phys. B **280**, 523 (1987).
 - [18] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
 - [19] P. A. Lee and N. Nagaosa, Phys. Rev. B **46**, 5621 (1992).
 - [20] A.M. Polyakov, Nucl. Phys. B **120**, 429 (1977). A.M. Polyakov, *Gauge fields and strings* (Harwood Academic Publishers, London, 1987).
 - [21] B. Halperin, Phys. Rev. B **25**, 2185 (1982).
 - [22] X. G. Wen, Phys. Rev. B **40**, 7387 (1989); Int. J. Mod. Phys. B **2**, 239 (1990).
 - [23] Congjun Wu, Phys. Rev. Lett. **101**, 186807 (2008).
 - [24] L. B. Shao, S. L. Zhu, L. Sheng, D. Y. Xing, Z. D. Wang, Phys. Rev. Lett. **101**, 246810 (2008).